ISO Presentation: A Domain Theoretic Approach to Computational Geometry

Sam Clegg
sbc02@doc.ac.ac.uk

March 2003

A Domain Theoretic Approach to Computational Geometry

- Study of a new approach to solid modelling and computational geometry.
- Encompassing several fields of theoretical computer science:
  - Computation Geometry
  - Topology
  - Computability Theory
  - Domain Theory
- Based largely on the work of Abbas Edalat and Andre Lieutoer.

Overview of Presentation

- Motivation for a new model.
- Topology.
- Domain Theory.
- The Solid Domain.
- Algorithms and data-types.
- Convex Hull.

Imperial College London

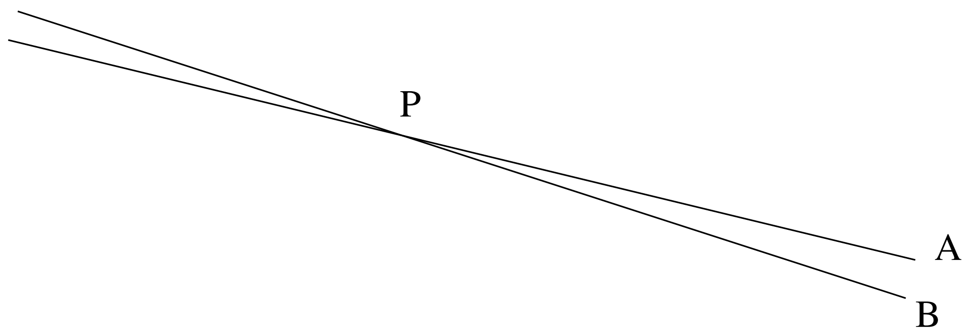
Why a new model?

- Algorithms from computation geometry are used throughout the design and manufacturing industries (CAD/CAM).
  - Most implementations model solids as the interior of a set of polygons with floating point (FP) vertices.
  - Algorithms rely on FP arithmetic, and the comparison of real numbers.
  - FP arithmetic is susceptible to numerical inaccuracies
    - Can lead to algorithmic errors.
  - Comparison of real numbers is undecidable!
    - Leads to basic predicates being non-computable.

- A new model and data-type is needed which can be used to create robust algorithms.

Algorithmic Errors Caused by FP

- Finding the intersection of two lines:



- The calculated P should lie on both lines but it many cases an FP algorithm will produce a point whose distance from A or B is non-zero.

Imperial College London

Non-computable Predicates

- Comparison of real numbers is not decidable and not computable
  - e.g. for $x \in \mathbb{R}$ does $x = 0$?
  - How many decimal zeros would been needed verify this?

- The membership predicate for any proper subset of $\mathbb{R}^3$ is not continuous and non-computable.

$$x \in A = \begin{cases} \mathrm{tt} & \text{if } x \in A \\ \mathrm{ff} & \text{if } x \notin A \end{cases}$$

  - If x lies on the boundary of A then no finite approximation to x will tell us weather it is inside or outside the set.

Euclidean Topology

- Real world object live Euclidean space ($\mathbb{R}^n$) so we consider only the induced Euclidean topology.

- When dealing with points and sets in $\mathbb{R}^n$ we define the following terms:
  - Neighbourhood
    - A neighbourhood of a point $x \in \mathbb{R}^n$ is any n-ball with centre x and radius > 0.
  - Boundary
    - The boundary ∂A of a set A is the set of points for which all neighbourhoods intersect both A and A^c.

Imperial College London

Euclidean Topology

- Open Sets
  - A set is open if every point has a neighbourhood lying in the set.
  - An open set includes none of its boundary points. (e.g. $(0,1)$)
  - The interior A° of a set A is its largest open subset.

- Closed Sets
  - A set is closed if it the complement of an open set.
  - A closed set includes all its boundary points. (e.g. $[0,1]$)
  - The closure $\overline{A}$ of a set A is its smallest closed superset.

Requirements for a new model

1. The notion of computability of solids has to be well defined.
2. It has to reflect the observable properties of real solids.
3. It must be closed under boolean operations and all basic predicates and operations have to be computable.
4. Non-regular sets have to be captured by the model as well as regular solids.
5. It has to support a design methodology for actual robust algorithms.

Domain Theory

- Domain theory is a tool for modelling approximations to uncountable sets, sequences, or computations.
- The term *domain* usually refers to Complete Partial Order.
- Partial Ordering
  - A *partial order* comprises a set and an ordering relation: $(A, \sqsubseteq)$
  - The relation must be reflexive, transitive and antisymmetric.
  - The ordering relation is known as an information ordering.
    If $x \sqsubseteq y$ we write:
    * x approximates y
    * y is a refinement of x

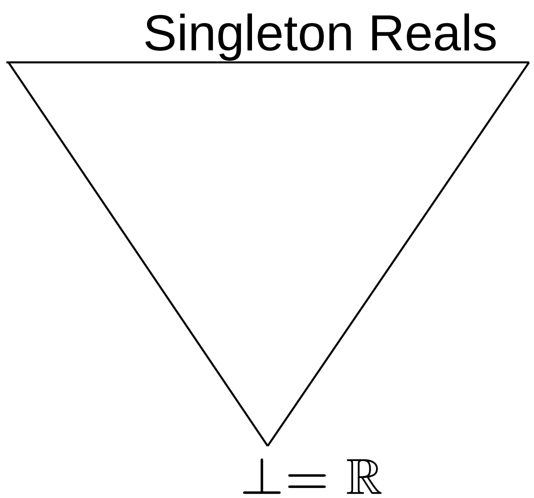
Domain Theory 2

- Complete Partial Order
  - Every chain has a least upper bound ($\bigsqcup$) within A.
  - There exists a unique bottom element ($\perp$) such that: $\perp \sqsubseteq x$ for all $x \in A$
- Example: The Interval Domain
  - A domain composed of closed intervals of $\mathbb{R}$: $\mathbf{I}\mathbb{R} = \{[a,b] | a, b \in \mathbb{R}\}$.
  - Intervals are ordered by reverse inclusion.

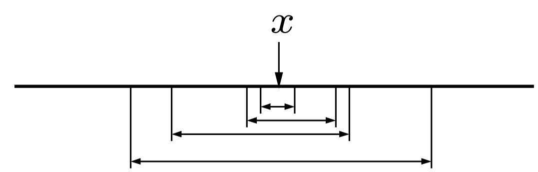
$$[a,b] \sqsubseteq [c,d] \iff [a,b] \supseteq [c,d]$$

  - Take $\perp = \mathbb{R}$ which includes all intervals.

The Interval Domain



The $\mathbf{I}\mathbb{R}$ domain.



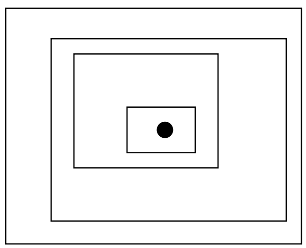
A chain in $\mathbf{I}\mathbb{R}$.

- Maximal elements are the Real numbers.
- Intervals can be thought of as *Partial Points*.
- This is a CPO: $\bigsqcup_{i\in n}[a_i, b_i] = \bigcap_{i\in n}[a_i, b_i]$

Partial Points

- The notion of partial points can be extended to d dimensions.

- The set $\mathbf{I}\mathbb{R}^d$ of closed d-rectangles ordered by reverse inclusion forms the domain of partial points in d.
  - For $d = 2$ a chain is a shrinking sequence of rectangles.
  - Maximal elements are points on the Real plane.
  - $\perp = \mathbb{R}^2$



Computability of Partial Points

- Basis
  A basis B for a domain D is a subset such that each $x \in D$ exists as the lub of all basis elements way-below x.

- ω-continuous domains
  As domain is ω-continuous if it has a countable basis.

- The set of closed rational rectangles $\mathbf{I}\mathbb{Q}^2$ form a basis for $\mathbf{I}\mathbb{R}^2$.

- $\mathbf{I}\mathbb{Q}^2$ is countable, hence $\mathbf{I}\mathbb{R}^2$ is ω-continuous

- An element of $x \in \mathbf{I}\mathbb{R}^2$ is *computable* if there exists total recursive function that enumerates the chain of basis elements with lub x.

A new model: Partial Solids

- Partial Solids composed of as a pair of *disjoint open* sets (I, E) representing the interior and exterior.

- Membership of an open set is *semi-decidable*.
  - If A is open and $x \in A$ we can determine this in finite time.
  - Converse is not true (specifically for $x \in \partial A$).

- Since I and E are both open we can give a continuous membership predicate for $x \in \mathbb{R}^n$ and $(I, E) \in \mathbf{S}\mathbb{R}^d$:

$$x \in (I, E) = \begin{cases} \mathrm{tt} & \text{if } x \in I \\ \mathrm{ff} & \text{if } x \in E \\ \perp & \text{otherwise} \end{cases}$$

Partial Solids

- We use $\perp$ represent elements for which membership non-observable.
- It can also represent elements for which membership is not yet known at some finite stage of computation.
- This allows *partially computed* or *partially input* solids to be modelled in data-types and algorithms.

The Solid Domain

- The Solid Domain is induced by the following ordering relation on $\mathbf{S}\mathbb{R}^d$:
$$(I_1, E_1) \sqsubseteq (I_2, E_2) \iff I_1 \subseteq I_2 \text{ and } E_1 \subseteq E_2$$

- A Partial Solids with a larger interior and/or exterior is a refinements of a smaller one.

- A smaller interior or exterior approximated a lager one.

- Clearly $\bot = (\emptyset, \emptyset)$

- Every chain has a lub in $\mathbf{S}\mathbb{R}^d$: $\bigsqcup_{i \in n}(I_i, E_i) = (\bigcup_{i \in n} I_i, \bigcup_{i \in n} E_i)$

Properties of the Solid Domain

- **Theorem:** Elements are maximal iff $I = E^{c\circ}$ and $E = I^{c\circ}$.
- All maximal elements have regular interior and exterior.
  - Since $I = E^{c\circ}$ and E^c and is closed.
- For any regular open set A there exists a maximal element $(A, A^{c\circ}) \in \mathbf{S}\mathbb{R}^d$.
- A partial solid is a *classical* solid object if

$$\overline{I} \cup \overline{E} = \mathbb{R}^n$$

- **Theorem:** All maximal elements are classical solids.

Computability of Partial Solids

- The solid domain is ω-continuous.
  - The set of partial rational polyhedra $\mathbf{S}\mathbb{Q}^d$ form a basis.
    $\mathbf{RP}=$ set of open rational polyhedra.
    $\mathbf{S}\mathbb{Q}^d = \{(I,E) | I,E \in \mathbf{RP}\}$
  - Any element of $\mathbf{S}\mathbb{R}^d$ is a lub of a chain from $\mathbf{S}\mathbb{Q}^d$.
  - Partial rational polyhedra are countable and computable.

- An element of $(I,E) \in \mathbf{S}\mathbb{R}^d$ is *computable* if there exists an effective enumeration of basis elements with lub (I,E).

Algorithms and Data-types

- Data-types:
  - Partial Solids
  - Partial Points
  - Partial Lines
  - Partial Circles

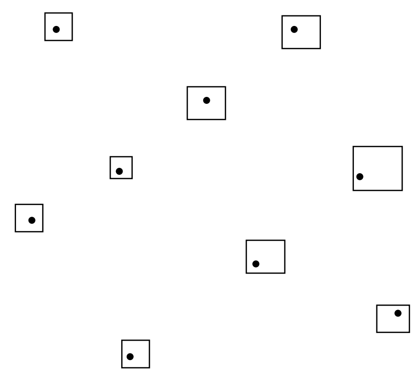
- Algorithms:
  - Convex Hull
  - Delauney Triangulation
  - Voronio Diagram

Convex Hull

- No classical algorithm in known for computing the convex hull on exact real inputs.

- **Goal:** Compute the convex hull of n Partial Points a Partial Solid.

$$C_n : (\mathbf{I}\mathbb{R}^2)^n \to \mathbf{S}\mathbb{R}^2$$



Convex Hull 2

- Using the classic hull algorithm (H_n) the Partial Convex Hull is defined as:
- $C_n(R_1, \cdots, R_n) = (I_n(R_1, \cdots, R_n), E_n(R_1, \cdots, R_n))$

$$I_n(R_1, \cdots, R_n) = \Big(\bigcap_{p_i \in R_i (i=1,\cdots,n)} H(p_1, \cdots, p_n) \Big)^{\circ}$$

$$E_n(R_1, \cdots, R_n) = \Big(\bigcup_{p_i \in R_i (i=1,\cdots,n)} H(p_1, \cdots, p_n) \Big)^{c}$$

- If input points are refined to single points on the real plain we simply get the interior and exterior of their hull.

Computable Convex Hull

- For $d = 2$ each partial point R has four vertices $R_i^1, R_i^2, R_i^3, R_i^4$.

Theorem: For $d = 2$ the interior I_n and E_n can be computed as:

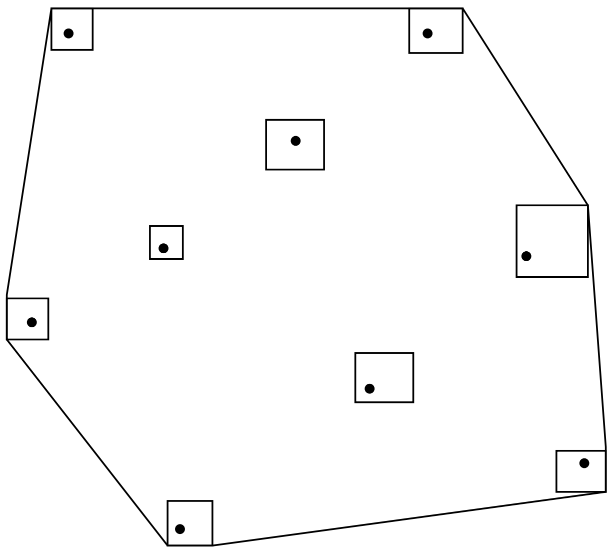
$$E_n(R_1, \cdots, R_n) = (H_{4n}(R_1^1, R_1^2, R_1^3, R_1^4, \cdots, R_n^1, R_n^2, R_n^3, R_n^4))^c$$

$$I_n(R_1, \cdots, R_n) = (\bigcap_{j=1}^{4} H_n(R_1^j, \cdots, R_n^j))^\circ$$

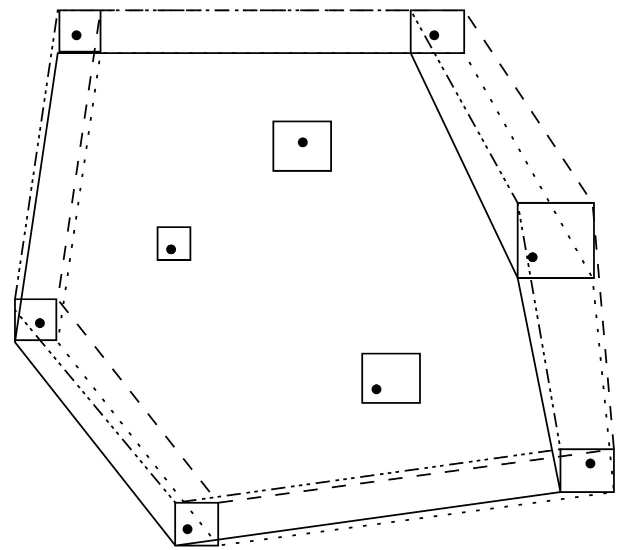
- Classical hull algorithm H_n is used but we can limit partial points to $\mathbf{I}\mathbb{Q}^2$ for which H_n is computable.

Computable Convex Hull

- Computable Hull Interior and Exterior



Hull exterior

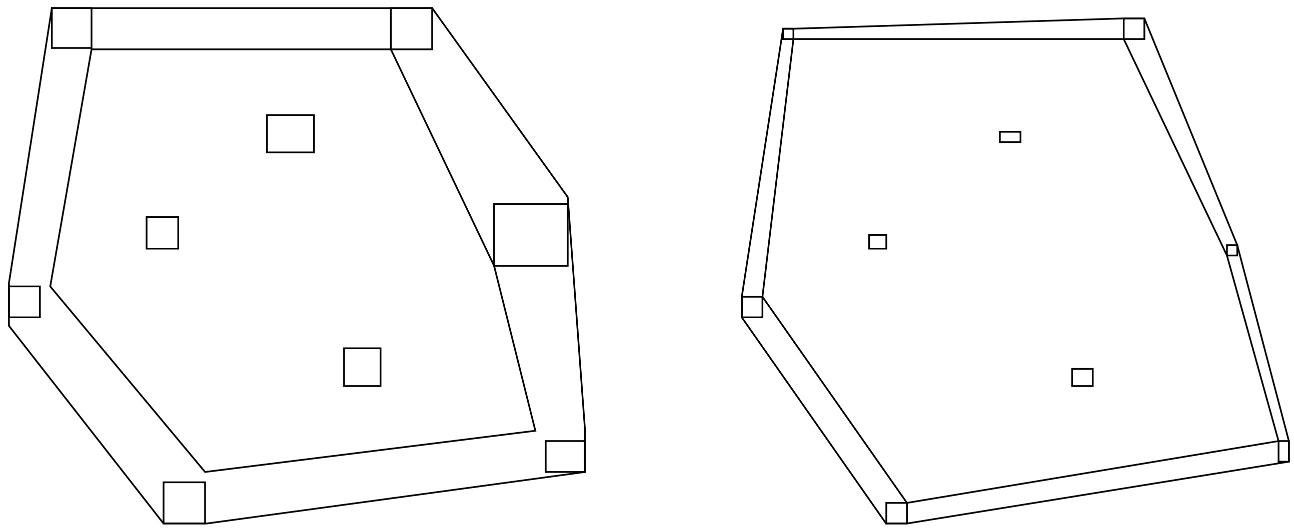


Hull Interior

Imperial College London

Convergence of Convex Hull

- As the partial points tend to points on the Real plain the convex hull tends to the classical solid object representing the hull of the points.



Conclusions

- Computability of Solids and other datatypes is well defined.
- Basic predicates and operations are continuous and computable.
- Framework exists for creating actual robust algorithms.

Imperial College London